MATH 245 F24, Exam 3 Solutions

1. Carefully define the following terms: intersection, nonempty.

   For any two sets S, T, their intersection is the set given by $\{x : x \in S \wedge x \in T\}$. A set S is nonempty if it does not equal the empty set (equivalently, if it contains at least one element).

2. Carefully define the following terms: inverse (relation), transitive.

   For any relation R from some set S to some set T, the inverse is the relation from T to S given by $\{(b, a) : (a, b) \in R\}$. A relation R on some set S is transitive if it satisfies: $\forall x, y, z \in S,\ (xRy \wedge yRz) \rightarrow (xRz)$.

3. Let $R = \{x \in \mathbb{Z} : x|24\}$, $S = \{x \in \mathbb{Z} : 2|x\}$, $T = \{x \in \mathbb{Z} : 3|x\}$. Prove or disprove that $R \subseteq S \cup T$.

   The statement is false, and requires a counterexample. There are exactly two to choose from: 1 and -1. Let's pick -1. We need to show that $-1 \in R$ and also that $-1 \notin S \cup T$.

   Note that $(-1)(-24) = 24$, and $-24 \in \mathbb{Z}$, so $-1 \in R$. Suppose now that $2k = -1$. Then $k = -0.5 \notin \mathbb{Z}$, so $2 \nmid -1$ and hence $-1 \notin S$. Suppose now that $3j = -1$. Then $j = -1/3 \notin \mathbb{Z}$, so $3 \nmid -1$ and hence $-1 \notin T$. By the conjunction semantic theorem, $(-1 \notin S) \wedge (-1 \notin T)$. By De Morgan's Law (for propositions), $\neg((-1 \in S) \vee (-1 \in T))$, and hence $\neg(-1 \in S \cup T)$ (definition of $\cup$). Lastly we conclude $-1 \notin S \cup T$, as desired.

   ALTERNATE SOLUTION: We first prove that $-1 \in R$ because $(-1)(-24) = 24$ and $-24 \in \mathbb{Z}$. Now, we continue by way of contradiction. Suppose that $-1 \in S \cup T$. Then $-1 \in S \vee -1 \in T$. We have two cases.
   Case $-1 \in S$: Then there is $k \in \mathbb{Z}$ with $2k = -1$, so $k = -0.5$. But $k \notin \mathbb{Z}$, so this case can't happen.
   Case $-1 \in T$: Then there is $j \in \mathbb{Z}$ with $3j = -1$, so $j = -1/3$. But $j \notin \mathbb{Z}$, so this case too can't happen.
   Hence neither case can happen! We have a contradiction, so $-1 \notin S \cup T$, as desired.

4. Let S, T be sets. Suppose that $S \Delta T = S \setminus T$. Prove that $T \subseteq S$.

   Let $x \in T$ be arbitrary. We will proceed, in two cases, to prove that $x \in S$.
   Case $x \in S$: Well that was easy, $x \in S$ already.
   Case $x \notin S$: By conjunction, $x \in T \wedge x \notin S$. By addition, $(x \in S \wedge x \notin T) \vee (x \in T \wedge x \notin S)$. Hence $x \in S \Delta T$. Since, by hypothesis, $S \Delta T = S \setminus T$, we have $x \in S \setminus T$. Hence $x \in S \wedge x \notin T$. By simplification, $x \in S$.

   In both cases, $x \in S$.

   ALTERNATE SECOND CASE: Case $x \notin S$: Proceed the same way, but get to a contradiction (e.g. $x \in S$ and $x \notin S$), so this second case cannot happen. Now the wrapup is "In the sole remaining case, $x \in S$" rather than "In both cases, $x \in S$".

   COMPLETELY DIFFERENT SOLUTION (found by a student): Argue by contradiction, suppose that $T \not\subseteq S$. Then there would be some $x \in T$ with $x \notin S$. By conjunction, $x \in T \wedge x \notin S$. By addition, $(x \in S \wedge x \notin T) \vee (x \in T \wedge x \notin S)$. Hence $x \in S \Delta T$. Since, by hypothesis, $S \Delta T = S \setminus T$, we have $x \in S \setminus T$. Hence $x \in S \wedge x \notin T$, and by simplification $x \in S$. This is a contradiction since we assumed that $x \notin S$.

5. Find a set S with $\left|S \cap 2^{S \times S}\right| = 2$. Give S carefully, in list notation.

   Note that $S \times S$ is a set containing exclusively ordered pairs, and $2^{S \times S}$ is a set containing exclusively (sets containing ordered pairs). Exactly two elements of S will need to also be elements of $2^{S \times S}$, so those two elements must be sets containing ordered pairs. Many solutions are possible.

   NORMAL SOLUTION: Take $S = \{a, b, \{(a, a)\}, \{(a, b), (b, b)\}\}$. Note that $a \in S$, so $(a, a) \in S \times S$, so $\{(a, a)\} \subseteq S \times S$, and thus $\{(a, a)\} \in 2^{S \times S}$. Similarly, $\{(a, b), (b, b)\} \in 2^{S \times S}$. These are the two elements in $S \cap 2^{S \times S}$.

   SLICK SOLUTION: Take $S = \{\emptyset, S \times S\}$. Each of $\emptyset$ and $S \times S$ are subsets of $S \times S$, so they are elements of $2^{S \times S}$. Hence in fact $S \subseteq 2^{S \times S}$ so $\left|S \cap 2^{S \times S}\right| = |S|$. However, the price of this slick solution is that now we need to show that $S \times S \neq \emptyset$, because otherwise $|S| = 1$, which is too small. This is done by noting that $(\emptyset, \emptyset) \in S \times S$, so $S \times S$ is nonempty.

6. Let S, T be sets with $T \subseteq S$. Let R be a trichotomous relation on S. Prove that $R|_T$ is trichotomous.

Let $x, y \in T$, and suppose that $(x, y) \notin R|_T$ and also $(y, x) \notin R|_T$. Since $T \subseteq S$, also $x, y \in S$. Since R is trichotomous, $x = y \vee (x, y) \in R \vee (y, x) \in R$, which gives three cases. However, case $(x, y) \in R$ cannot happen, because then $(x, y) \in R|_T$ (since $x, y \in T$ and $(x, y) \in R$), which we know not to be the case. Also, case $(y, x) \in R$ cannot happen either, because then, similarly, $(y, x) \in R|_T$, which we also know not to be the case. Hence, $x = y$.

ALTERNATE ENDING: Instead of three cases, we can use disjunctive syllogism twice to go from $x = y \vee ((x, y) \in R) \vee ((y, x) \in R)$, $(x, y) \notin R$, and $(y, x) \notin R$, to $x = y$.

7. R is a relation on $S = \{1, 2\}$, and $R \neq R_{empty}$. Prove or disprove: R is symmetric and antisymmetric, if and only if, $R = R_{diagonal}$.

The hardest part of this problem is understanding how all the words make sense together. The statement has two halves. One half is "If $R = R_{diagonal}$, then R is symmetric and antisymmetric". This half happens to be true, though it doesn't matter. The other half is "If R is symmetric and antisymmetric, then $R = R_{diagonal}$". This half is not true (hence the whole thing is not true), and requires a counterexample. A counterexample needs to be a relation on S that is symmetric, antisymmetric, but not equal to $R_{diagonal}$ or R_{empty}. A hint on how to find such a relation was Exercise 10.9, in which you proved that such a relation would be a subset of $R_{diagonal}$. Two counterexamples are possible: $\{(1, 1)\}$ and $\{(2, 2)\}$.

Let's prove that $R = \{(1, 1)\}$ is a counterexample. Note that $R \neq R_{empty}$ since $(1, 1) \in R$ and $(1, 1) \notin R_{empty}$. Also, $R \neq R_{diagonal}$ since $(2, 2) \in R_{diagonal}$ but $(2, 2) \notin R$. If $(x, y) \in R$, then $(x, y) = (1, 1)$, so $x = y = 1$, so also $(y, x) \in R$. This proves that R is symmetric. If instead $xRy \wedge yRx$, then again $(x, y) = (1, 1)$ so $x = y = 1$, and in particular $x = y$. This proves that R is antisymmetric.

8. Let S, U be sets with $S \subseteq U$. Prove that $S \cup S^c = U$.
Note: This is part of Theorem 9.2. Do not use the theorem to prove itself!

Proving $S \cup S^c \subseteq U$: Let $x \in S \cup S^c$. Then $x \in S \vee x \in S^c$. Two cases. If $x \in S$ then (since $S \subseteq U$) $x \in U$. If instead $x \in S^c$ then $x \in U \setminus S$ so $x \in U \wedge x \notin S$, so by simplification $x \in U$. In both cases $x \in U$.
Proving $U \subseteq S \cup S^c$: Let $x \in U$. Two cases. If $x \in S$ then by addition $x \in S \vee x \in S^c$, so $x \in S \cup S^c$. If instead $x \notin S$ then by conjunction $x \in U \wedge x \notin S$, so $x \in U \setminus S$, so $x \in S^c$. Now, by addition $x \in S \vee x \in S^c$, so $x \in S \cup S^c$. In both cases, $x \in S \cup S^c$.

9. Let S be a set with relation R. Let R_2 be the reflexive closure of R. Suppose R_1 satisfies $R \subseteq R_1 \subseteq R_2$. Suppose also that R_1 is reflexive (on S). Prove that $R_1 = R_2$.

$R_1 \subseteq R_2$ is part of the hypothesis, so it remains to prove that $R_2 \subseteq R_1$. Let $x \in R_2$, and so $x \in R \cup \{(a, a) : a \in S\}$. Hence $x \in R \vee x \in \{(a, a) : a \in S\}$. Two cases.
Case $x \in R$: Since $R \subseteq R_1$, also $x \in R_1$.
Case $x \in \{(a, a) : a \in S\}$: So, $x = (a, a)$ for some $a \in S$. Since R_1 is reflexive, and $a \in S$, in fact $(a, a) \in R_1$. Hence, $x \in R_1$.

In both cases, $x \in R_1$.

NOTE: This problem proves that the reflexive closure is parsimonious, the "smallest possible" reflexive relation that contains R.

10. Set $S = \mathbb{R} \setminus \{0\}$. Prove that $|S| = |\mathbb{R}|$.

We need a pairing between S and $\mathbb{R}$. Most elements should be paired with themselves, since $S, \mathbb{R}$ are almost the same set. Here is one possible pairing, the most natural one:

For every $a \in S$, we pair $a \leftrightarrow b$, where $b = \begin{Bmatrix} a - 1 & a \in \mathbb{N} \\ a & a \notin \mathbb{N} \end{Bmatrix} \in \mathbb{R}$.

Any pairing that works is good enough, but here's an explanation if the pairing alone doesn't make sense: Set $T = \mathbb{R} \setminus \mathbb{N}_0$, and note that $S = T \cup \mathbb{N}$ while $\mathbb{R} = T \cup \mathbb{N}_0$. We pair everything in T with itself, i.e. if $x \in T$ then $x \leftrightarrow x$. What's left is to pair $\mathbb{N}$ with $\mathbb{N}_0$, and we've done this already in class (it was the first example with equicardinal infinite sets), pairing $x \leftrightarrow x - 1$.